Comments of the definitions of coherent states for the SUSY harmonic oscillator

1993 J. Phys. A: Math. Gen. 26 6271

(http://iopscience.iop.org/0305-4470/26/22/026)

J. Phys. A: Math. Gen. 26 (1993) 6271–6275. Printed in the UK

Comments of the definitions of coherent states for the SUSY harmonic oscillator

Y Bérubé-Lauzière and V Hussin

Centre de recherches mathématiques, Université de Montréal C.P. 6128-A, Montréal (Québec), H3C 3J7, Canada

Received 10 May 1993

Abstract. We want to insist on the possibility of regarding coherent states for the supersymmetric harmonic oscillator equivalently in three different ways, just as in the usual bosonic case.

As has already been mentioned [1], coherent states can be defined in three different ways: as minimum uncertainty states, as eigenstates of an annihilation operator or as displacement operator coherent states. This last approach may be described in other words as a group–theoretical treatment [2] of coherent states. These definitions appear naturally in the most simple example of the harmonic oscillator and in this case they coincide. It must be noticed that coherent states and their applications [3] have received much attention in the different directions given above but, apart from the example of the harmonic oscillator, they give, in general, non-equivalent results.

Starting once again with the usual harmonic oscillator, which will be called bosonic in the following, an extension has been considered to involve both bosonic and fermionic degrees of freedom. It is of course the supersymmetric (SUSY) harmonic oscillator.

In its quantum version it has been initiated by Witten [4] and has been the starting point for several group–theoretical considerations, such as the existence of invariance superalgebras and supergroups [5].

This SUSY harmonic oscillator has been the example to consider in order to introduce the concept of supercoherent states which has been developed, as one might expect, in the three ways given above. In fact, the first appearance of such states came from Aragone and Zypman [1], where they are defined as eigenstates of an annihilation operator. Later, it was shown [6] that it is not possible to relate them to minimum uncertainty states and to a group–theoretical approach. Independently, the group–theoretical point of view has been treated from a generalization of Perelomov's approach [2] based on supergroups and their representations [7, 8].

A point that has already been noticed [8] is the fact that the SUSY harmonic oscillator supercoherent states do not share with their usual (non-super) counterparts the equivalence in their defining properties. In fact, we want to show here that it is mainly the definition of an annihilation operator given in [1] that causes all the trouble. So we will suggest another definition in order to clarify the connection between the three approaches.

0305-4470/93/226271+05 $07.50

Let us start with the usual considerations in the quantum SUSY harmonic oscillator. It is characterized by the Hamiltonian

$$H=\tfrac{1}{2}(\hat{p}^2+\omega^2\hat{q}^2)+\tfrac{1}{2}\omega\sigma_3=\omega(a^+a+f^+f) \tag{1}$$

where we have introduced the bosonic (a,a^+) and fermionic (f,f^+) creation and annihilition operators. They are defined as usual by

$$a^+=\frac{1}{\sqrt{2\omega}}(\omega\hat{q}-\mathrm{i}\hat{p}) \qquad a=\frac{1}{\sqrt{2\omega}}(\omega\hat{q}+\mathrm{i}\hat{p}) \tag{2}$$

$$f^+=\sigma_+=\tfrac{1}{2}(\sigma_1+\mathrm{i}\sigma_2) \qquad f=\sigma_-=\tfrac{1}{2}(\sigma_1-\mathrm{i}\sigma_2) \tag{3}$$

(the σ_i's being the usual Pauli matrices) and satisfy the well known relations

$$[a,a^+]=1 \qquad \{f,f^+\}=1. \tag{4}$$

Note that its SUSY character appears from the fact that the Hamiltonian can be written as the anti-commutator of the conserved supercharges Q and Q^+:

$$H=\{Q,Q^+\} \qquad Q=\mathrm{i}\sqrt{\omega}a^+f \qquad Q^+=\mathrm{i}\sqrt{\omega}af^+. \tag{5}$$

Since we are concerned with characterization of supercoherent states in different ways, it is necessary to give the representation space on which they will be defined. The natural choice is the Fock space

$$\mathscr{F}=\mathscr{F}_b\otimes\mathscr{F}_f=\{|e_0^n\rangle_b=|n,0\rangle, |e_1^n\rangle_f=|n,1\rangle, n=0,1,2\ldots\} \tag{6}$$

with the energy eigenvectors as basis vectors. The fermionic sector is generated by $|e_1^n\rangle_f$ while the bosonic one by $|e_0^n\rangle_b$ for all values of n, since the fermionic number operator $N_f=f^+f$ acts on these states as follows

$$N_f|e_1^n\rangle_f=|e_1^n\rangle_f, \qquad N_f|e_0^n\rangle_b=0. \tag{7}$$

Following Aragone and Zypman [1], supercoherent states may be constructed as eigenstates of the supersymmetric annihilation operator

$$A=\begin{pmatrix} a & 0 \\ 1 & a \end{pmatrix} \tag{8}$$

which satisfies

$$[H,A]=-\omega A. \tag{9}$$

They form a two-dimensional subspace of $\mathscr{F}$ and an arbitrary state is given by

$$|Z\rangle=\mathrm{e}^{-|z|^2/2}[\cos\theta|z\rangle_b+\sin\theta\,\mathrm{e}^{\mathrm{i}\varphi}|z\rangle_s]. \tag{10}$$

The state $|z\rangle_b$ refers to the usual bosonic coherent state and is explicitly given by

$$|z\rangle_b=|z,0\rangle=\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}|e_0^n\rangle_b. \tag{11}$$

If we introduce the notation $|z\rangle_f$ for the corresponding purely fermionic state, i.e.

$$|z\rangle_f=|z,1\rangle=\sum_{n=0}\frac{z^n}{\sqrt{n!}}|e_1^{n+1}\rangle_f \tag{12}$$

we can write the SUSY coherent state $|z\rangle_s$ as

$$|z\rangle_s = \frac{1}{\sqrt{2}}(|z\rangle_f + \bar{z}|z\rangle_b - |z'\rangle_b) \tag{13}$$

where

$$|z'\rangle_b = \frac{\mathrm{d}}{\mathrm{d}z}|z\rangle_b = a^+|z\rangle_b.$$

We clearly have

$$A|z\rangle_b = z|z\rangle_b \qquad A|z\rangle_s = z|z\rangle_s \tag{14}$$

and the relations

$${}_b\langle z|z\rangle_s = 0 \qquad {}_b\langle z|z\rangle_b = {}_f\langle z|z\rangle_f = {}_s\langle z|z\rangle_s = \mathrm{e}^{|z|^2}. \tag{15}$$

Note that the results are slightly different from [1] because of the choice of the Hamiltonian in (1). This way, the purely bosonic (purely fermionic in [1]) coherent state will be recovered as a particular coherent state.

Now we can see that the operator

$$\tilde{A} = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \tag{16}$$

is also a good SUSY annihilation operator because it satisfies the commutation relation (9). It is nothing other than an amplification of the bosonic annihilation operator. So the normalized state $|\tilde{Z}\rangle$ given by

$$|\tilde{Z}\rangle = \mathrm{e}^{-|z|^2/2}(\cos\theta|z\rangle_b + \sin\theta\,\mathrm{e}^{i\phi}|z\rangle_f) \tag{17}$$

is another candidate for a supercoherent state, defined as an eigenstate of $\tilde{A}$. Putting $v = -\tan\theta\,\mathrm{e}^{i\phi}$ and assuming $\cos\theta > 0$, it takes the form

$$|\tilde{Z}\rangle = \mathrm{e}^{-|z|^2/2}\frac{1}{(1+\bar{v}v)^{1/2}}(|z\rangle_b - v|z\rangle_f) \tag{18}$$

which will be of interest for the following. While such a state seems to be the natural choice to remain very close to the purely bosonic theory, it has not been considered in [1].

We have then two types of supercoherent states and we propose to show why the last candidate (18) will play a more significant role. In fact, we will show that it can be connected to the definitions of such states based on group theory and on a superclassical approach.

The group-theoretical approach, which is based here on the Weyl–Heisenberg supergroup W, has been treated in [8]. For the superclassical approach, while some information may be found in the literature about bosonic and fermionic oscillator coherent states [9, 10], the SUSY version has not been completely investigated. This is what we propose to do here in a way that makes clear the connection with the group-theoretical approach.

We first summarize the results for the supercoherent states constructed from the action of a unitary representation of W. Such a representation is given by

$$T(g) = \exp\{\alpha a^+ - \bar{\alpha}a + \mathrm{i}dI + \eta f^+ + \bar{\eta}f\} \tag{19}$$

where d, α are c-numbers (even Grassmann numbers) and η is an a-number (odd Grassmann number) [9]. Note that a, a^+, f, f^+ and I that satisfy (4), generate the

Weyl–Heisenberg superalgebra $\mathcal{W}$. From this point of view, a supercoherent state is given by

$$|\alpha,\eta\rangle_{\rm GT}=T(g)|e_0^0\rangle_b \tag{20}$$

which is a displacement of the ground state, $T(g)$ being called the displacement operator.

Using the factorization

$$T(g)=\mathrm{e}^{(\mathrm{i}d-(\bar\eta\eta+|\alpha|^2)/2)I}\,\mathrm{e}^{\alpha a^+}\,\mathrm{e}^{-\eta f^+}\,\mathrm{e}^{-\bar\alpha a}\,\mathrm{e}^{\bar\eta f} \tag{21}$$

and the fact that a and f act trivially on $|e_0^0\rangle_b$, we get

$$\begin{aligned}|\alpha,\eta\rangle_{\rm GT}&=\mathrm{e}^{-\bar\eta\eta/2}\,\mathrm{e}^{-|\alpha|^2/2}(|\alpha,0\rangle-\eta|\alpha,1\rangle)\\&=\left(1-\frac{\bar\eta\eta}{2}\right)\mathrm{e}^{-|\alpha|^2/2}(|\alpha,0\rangle-\eta|\alpha,1\rangle)\end{aligned} \tag{22}$$

where we have omitted the phase factor $\mathrm{e}^{\mathrm{i}d}$ and taken into account the a-type character of $\eta(\eta^2=0)$. Let us insist on the fact that the state (22) is an eigenstate of both the bosonic and fermionic annihilation operators. Indeed, we have

$$a|\alpha,\eta\rangle_{\rm GT}=\alpha|\alpha,\eta\rangle_{\rm GT},\qquad f|\alpha,\eta\rangle_{\rm GT}=\eta|\alpha,\eta\rangle_{\rm GT}. \tag{23}$$

For the second equality we have explicitly taken into account the nilpotent nature of η. In connection with the type of numbers we are considering, let us also give a type to the vectors: $|\alpha,0\rangle$ will be of c-type and $|\alpha,1\rangle$ will be of a-type so that the supercoherent state (22) is of c-type. We follow here the convention of De Witt [9].

We will now show that the superclassical approach will give an interpretation of the state (22) closest to the superclassical one. In this approach we require the evolution of the mean values of the quantum operators over the coherent states to be the same as the evolution of the superclassical dynamical variables. The superclassical Hamiltonian for the SUSY harmonic oscillator is given by [10]

$$\mathcal{H}=\tfrac{1}{2}(p_q^2+\omega^2q^2)-\frac{\mathrm{i}\omega}{2}(p_\psi\psi-\psi p_\psi) \tag{24}$$

where q and ψ appear, respectively, as the c- and a-type dynamical variables and p_q and p_ψ are the corresponding conjugated variables. The Hamilton equations give

$$\dot\alpha(t)=-\mathrm{i}\omega\alpha(t)\qquad \alpha=\frac{1}{\sqrt{2\omega}}(\omega q+\mathrm{i}p) \tag{25}$$

and

$$\dot\psi(t)=-\mathrm{i}\omega\psi(t)\qquad \dot p_\psi(t)=\mathrm{i}\omega p_\psi(t). \tag{26}$$

The superclassical solutions are

$$\alpha(t)=\alpha\mathrm{e}^{-\mathrm{i}\omega t} \tag{27}$$

and

$$\psi(t)=\eta\,\mathrm{e}^{-\mathrm{i}\omega t}\qquad p_\psi(t)=\mathrm{i}\bar\eta\,\mathrm{e}^{\mathrm{i}\omega t} \tag{28}$$

where α is now a c-number and η an a-number. The classical energy is

$$\mathcal{H}_{\rm cl}=\omega(|\alpha|^2+\eta\bar\eta). \tag{29}$$

As is well known [10], the quantization associates with the classical variables α and ψ the operators a and f. So if we denote by $|C.S\rangle$ the state we are searching for, the mean values over this state of the quantum operators a, f and H must satisfy

$$\langle C.S|a|C.S\rangle=\alpha \qquad \langle C.S|f|C.S\rangle=\eta \qquad \langle C.S|H|C.S\rangle=\omega(|\alpha|^2+\eta\bar{\eta}). \tag{30}$$

It is then easy to prove, using (30), that $\mathscr{A}=a-\alpha$ and $\mathscr{N}=f-\eta$ satisfy

$$\|\mathscr{A}|C.S\rangle\|^2+\|\mathscr{N}|C.S\rangle\|^2=0\Leftrightarrow\|\mathscr{A}|C.S\rangle\|=\|\mathscr{N}|C.S\rangle\|=0 \tag{31}$$

and so

$$a|C.S\rangle=\alpha|C.S\rangle \qquad f|C.S\rangle=\eta|C.S\rangle. \tag{32}$$

Solving simultaneously these two eigenvalues equations, we obtain for $|C.S\rangle$, once normalized, the same result as for the group-theoretical approach. Note also that the superclassical approach gives a meaning to the parameters α and η appearing in the $|C.S\rangle$ in terms of the superclassical dynamical variables q and ψ; they are the initial values of the latter (cf. (28)). Such an interpretation cannot be obtained with the more abstract group-theoretical approach.

With these last results we have the connection between the three approaches to supercoherent states. Indeed, it suffices to notice the analogy between (18) and (22). The identification $\alpha=z$ is immediate because α is of c-type and may be an ordinary complex number. For the correspondence $v\sim\eta$, it is more subtle. It is all the quantity $\eta|\alpha, 1\rangle$ (which is a c-type vector) which will be identified to $v|z\rangle_f$.

Acknowledgments

The authors thank Luis Miguel Nieto for a careful reading of the manuscript. Y B L acknowledges a master's degree fellowship from NSERC of Canada. The research of VH is partially supported by research grants from NSERC of Canada and FCAR du Gouvernement du Québec.

References

[1] Aragone C and Zypman F 1986 *J. Phys. A: Math. Gen.* **19** 2267
[2] Perelomov A M 1986 *Generalized Coherent States and their Applications* (Berlin: Springer)
[3] Klauder J R and Skagerstam B (eds) 1985 *Coherent States: Applications in Physics and Mathematical Physics* (Singapore: World Scientific)
[4] Witten E, 1981 *Nucl. Phys.* B **188** 513
[5] de Crombrugghe M and Rittenberg V 1983 *Ann. Phys.* **151** 99
Balantekin A B 1985 *Ann. Phys.* **164** 277
Kostelecky V A, Nieto M M and Truax D R 1985 *Phys. Rev.* D **32** 2627
Beckers J, Dehin D and Hussin V 1987 *J. Phys. A: Math. Gen.* **20** 1137; 1988 **21** 651
[6] Orszag M and Salamó S 1988 *J. Phys. A: Math. Gen.* **21** L1059
[7] Balantekin A B, Schmitt H A and Barrett B R 1988 *J. Math. Phys.* **29** 1634
[8] Fatyga B W, Kostelecky V A and Truax D R 1991 *Phys. Rev.* D**43** 1403
[9] De Witt B 1992 *Supermanifolds* (Cambridge: Cambridge University Press)
[10] Ravndal F, *Proceedings of 1984 CERN School of Physics* Lofthus Norway p 300